Nature of Roots: The Discriminant

Name: Class / Tutor: Date:

Learning objectives. By the end of this lesson, you should be able to:

1. use the discriminant to determine the nature of the roots of a quadratic equation without solving it;
2. find an unknown constant in a quadratic equation from a given condition on its roots.

Recall · Last lesson: Solving quadratic equations

A **quadratic equation** in x can always be written in the **general form** $ax^2 + bx + c = 0$, where $a \neq 0$.

Method 1 · Factorisation e.g. $x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3$ or $x = -2$

Method 2 · Quadratic formula works for every quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Warm-up. Solve each equation. Leave answers in surd form where necessary.

R1. $x^2 - 5x + 6 = 0$

R2. $2x^2 + 3x - 4 = 0$

R3. $x^2 + 6x + 9 = 0$

R4. $x^2 + 2x + 5 = 0$

Think. R1 and R2 each gave two different answers, R3 gave only one, and R4 gave none. Look at the value of $b^2 - 4ac$ under the square root sign in each question. What do you notice?

Key Point 1 · The discriminant

For $ax^2 + bx + c = 0$, the expression under the square root in the quadratic formula is called the **discriminant**, written Δ (read "delta"):

$$\Delta = b^2 - 4ac$$

The **sign** of Δ tells us the nature of the roots, *without solving the equation*:

Value of Δ	Nature of roots	From the Recall
$\Delta > 0$	two **distinct** (different) real roots	R1: $\Delta = 1$ R2: $\Delta = 41$
$\Delta = 0$	two **equal** real roots (a repeated root): $x = \dfrac{-b}{2a}$	R3: $\Delta = 0$
$\Delta < 0$	**no** real roots	R4: $\Delta = -16$

Why? If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number. If $\Delta = 0$, "$\pm\sqrt{0}$" gives the same answer twice.

Common mistakes

- **Not rearranging first.** In $x^2 = 3x - 4$, b is *not* 3. Rewrite as $x^2 - 3x + 4 = 0$, so $a = 1$, $b = -3$, $c = 4$.
- **Sign errors.** If $b = -3$, then $b^2 = (-3)^2 = 9$, not -9.
- **Mixing up the wording.** "Real roots" means $\Delta \geq 0$ (distinct *or* equal), not only $\Delta > 0$.

Example 1 Without solving, determine the nature of the roots of each equation.

Solution

(a) $3x^2 - 5x + 1 = 0$: $\Delta = (-5)^2 - 4(3)(1) = 25 - 12 = 13 > 0$ $\therefore$ two distinct real roots.

(b) $4x^2 + 4x + 1 = 0$: $\Delta = 4^2 - 4(4)(1) = 16 - 16 = 0$ $\therefore$ two equal real roots.

(c) $x^2 = 2x - 3$. First rewrite: $x^2 - 2x + 3 = 0$.
$\Delta = (-2)^2 - 4(1)(3) = 4 - 12 = -8 < 0$ $\therefore$ no real roots.

Example 2 Without solving, determine the nature of the roots of $(x + 2)^2 = 4x + 5$.

Solution

Expand and rearrange: $x^2 + 4x + 4 = 4x + 5 \Rightarrow x^2 - 1 = 0$

Here there is no x term, so $a = 1$, $b = 0$, $c = -1$.

$\Delta = 0^2 - 4(1)(-1) = 4 > 0$ $\therefore$ two distinct real roots.

Example 3 Show that the equation $x^2 + (k - 3)x + (1 - k) = 0$ has two distinct real roots for all real values of k.

Solution

$\Delta = (k - 3)^2 - 4(1)(1 - k) = k^2 - 6k + 9 - 4 + 4k = k^2 - 2k + 5$

Complete the square: $\Delta = (k - 1)^2 + 4$

Since $(k - 1)^2 \geq 0$ for all real k, we have $\Delta \geq 4 > 0$.

$\therefore$ The equation has two distinct real roots for all real values of k.

Practice A · Nature of roots

Without solving the equation, determine the nature of its roots. Show the value of Δ.

BASIC

A1. $2x^2 + 7x + 3 = 0$

A2. $4x^2 - 20x + 25 = 0$

A3. $3x^2 + x + 2 = 0$

A4. $0.2x^2 - 0.4x + 0.5 = 0$

STANDARD

A5. $(2x - 1)^2 = x + 3$

A6. $(x + 1)(x - 5) = 2x - 14$

A7. $\dfrac{x + 2}{x - 1} = x$

A8. $2x(x - 3) = (x - 2)(x - 4)$

Practice A · (continued)

ADVANCED

A9. Show that the equation $x^2 + (k-1)x - k = 0$ has real roots for all real values of k.

A10. Show that the equation $kx^2 - (k+3)x + 3 = 0$, where $k \neq 0$, has real roots for all non-zero values of k. For which value of k are the two roots equal?

A11. Show that the equation $(x-a)(x-b) = 1$ has two distinct real roots for all real values of a and b.

A12. Given that $x^2 + px + q = 0$ has two equal real roots, show that $x^2 + (p+2)x + (p+q+1) = 0$ also has two equal real roots.

A13. (Challenge) The equation $x^2 + bx + c = 0$ has no real roots. Show that $c > 0$.

Key Point 2 · Finding an unknown constant

If we are told the nature of the roots, we can turn the words into a **condition on** Δ, then solve for the unknown.

The question says the equation has…	Condition
two distinct real roots	$\Delta > 0$
two equal real roots / a repeated root / one (double) real root	$\Delta = 0$
no real roots	$\Delta < 0$
real roots	$\Delta \geq 0$

Four steps

1. Write the equation in general form
2. Find Δ in terms of the unknown
3. Write the condition on Δ
4. Solve the equation / inequality

DSE exam tip

- Always **write the condition** (e.g. "$\Delta = 0$") as a separate line. In long questions this step earns the method mark even if your arithmetic slips later.
- If the x^2 term contains the unknown (e.g. $kx^2 + 6x + 1 = 0$), remember $k \neq 0$, otherwise it is not a quadratic equation.
- In Paper 2 MC, this idea often appears with graphs: "the graph *touches* the x-axis" means $\Delta = 0$. We will meet this next lesson.

Example 4 The equation $x^2 + kx + 9 = 0$ has two equal real roots. Find the possible values of k.

Solution

$\Delta = k^2 - 4(1)(9) = k^2 - 36$

Two equal real roots: $\Delta = 0$

$k^2 - 36 = 0 \Rightarrow k^2 = 36 \Rightarrow k = 6$ or $k = -6$

Example 5 The equation $x^2 - 4x + (k + 1) = 0$ has no real roots. Find the range of values of k.

Solution

$\Delta = (-4)^2 - 4(1)(k + 1) = 16 - 4k - 4 = 12 - 4k$

No real roots: $\Delta < 0$

$12 - 4k < 0 \Rightarrow 12 < 4k \Rightarrow k > 3$

Example 6 The equation $(k + 1)x^2 - 4x + (k - 2) = 0$, where $k \neq -1$, has two equal real roots. Find the possible values of k.

Solution

$\Delta = (-4)^2 - 4(k + 1)(k - 2) = 16 - 4(k^2 - k - 2) = -4k^2 + 4k + 24$

Two equal real roots: $\Delta = 0$

$-4k^2 + 4k + 24 = 0 \Rightarrow k^2 - k - 6 = 0 \Rightarrow (k - 3)(k + 2) = 0$

$k = 3$ or $k = -2$ (both satisfy $k \neq -1$)

Practice B · Finding unknown constants

Write the condition on Δ clearly before solving.

BASIC

B1. $x^2 + kx + 16 = 0$ has two equal real roots. Find the possible values of k.

B2. $x^2 - 6x + 2k = 0$ has no real roots. Find the range of values of k.

STANDARD

B3. $x^2 - 2(k+1)x + (k^2+5) = 0$ has real roots. Find the range of values of k.

B4. $(k-1)x^2 + 2kx + (k+2) = 0$, where $k \neq 1$, has two equal real roots. Find k.

B5. $x^2 + (2k-1)x + k^2 = 0$ has no real roots. Find the range of values of k.

B6. $x^2 - (k+1)x + 4 = 0$ has two equal real roots. Find the possible values of k.

B7. $x^2 + kx + (k+3) = 0$ has two equal real roots. Find the possible values of k and the root in each case.

B8. $(k-2)x^2 + 4x + 1 = 0$ is a quadratic equation with real roots. Find the range of values of k.

Practice B · (continued)

ADVANCED

B9. The equation $x^2 + 4x + k = 0$ has two distinct real roots, but the equation $x^2 - 2x + (k + 3) = 0$ has no real roots.
(a) Find the range of values of k. (b) How many integer values can k take?

B10. The straight line $y = 2x + c$ and the curve $y = x^2 + 3$ meet at exactly one point. Find the value of c.
(Hint: at an intersection point, $x^2 + 3 = 2x + c$.)

B11. (Challenge) The perimeter of a rectangle is 20 cm. Let one side be x cm and the area be A cm^2.
(a) Show that $x^2 - 10x + A = 0$. (b) Using the discriminant, show that $A \leq 25$.
(c) Can the area of the rectangle be 30 cm^2? Explain your answer.

Homework · Nature of roots

Hand in at the start of the next lesson. Suggested time: 90 minutes. Total: 69 marks. Show your working clearly.

Section A(1) · Short questions (18 marks)

1. Without solving, determine the nature of the roots of each equation. (6 marks)

(a) $(3x-1)(x+2)=2x-5$ (b) $\dfrac{x}{2}+\dfrac{2}{x}=2$ (c) $(2x+1)^2=3(2x+1)$

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2. The equation $x^2+6x+3k=0$ has two equal real roots. Find the value of k. (3 marks)

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3. The equation $2x^2-3x+(k+1)=0$ has no real roots. Find the range of values of k. (3 marks)

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4. The equation $x^2-2(k-1)x+(k^2+3)=0$ has real roots. Find the range of values of k. (3 marks)

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5. Show that the equation $x^2-(k+2)x+(k-1)=0$ has two distinct real roots for all real values of k. (3 marks)

Section A(2) · Standard questions (20 marks)

6. The equation $(k+1)x^2 - 6x + (k-7) = 0$, where $k \neq -1$, has two equal real roots. Find the possible values of k. (4 marks)

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7. The equation $x^2 + 2kx + (5k-6) = 0$ has two equal real roots. (5 marks)
(a) Find the possible values of k. (b) For each value of k in (a), find the root of the equation.

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8. The equation $x^2 - (2k+1)x + (k^2+2) = 0$ has two distinct real roots. (4 marks)
(a) Find the range of values of k. (b) Find the least integer value of k.

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9. Show that the equation $kx^2 + (2k+1)x + (k+1) = 0$, where $k \neq 0$, has two distinct real roots for all non-zero values of k. (3 marks)

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10. Is there a value of k such that $x^2 + 2x + k = 0$ has two distinct real roots **and** $x^2 - 4x + 2k = 0$ has no real roots? Explain your answer. (4 marks)

Section B · Long questions (17 marks)

11. The straight line $y = kx - 1$ meets the curve $y = x^2 + 3$ at exactly one point. (6 marks)

(a) Show that $x^2 - kx + 4 = 0$. (b) Find the possible values of k.

(c) For each value of k in (b), find the coordinates of the point where the line meets the curve.

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12. A piece of wire 40 cm long is bent to form a rectangle. Let the length of one side be x cm and the area be A cm^2. (6 marks)

(a) Show that $x^2 - 20x + A = 0$. (b) Using the discriminant, show that $A \le 100$.

(c) Is it possible for the area to be 96 cm^2? If yes, find the dimensions of the rectangle. If no, explain why.

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13. **(Challenge)** The equations $x^2 + 2ax + b = 0$ and $x^2 + 2bx + a = 0$ both have two equal real roots, where a and b are real numbers. Find all possible pairs (a, b). (5 marks)

Section C · Multiple choice (Circle one answer. 2 marks each, 14 marks.)

14. If the equation $x^2 - 2kx + k + 2 = 0$ has two equal real roots, then $k =$

A. -1 or 2
B. -2 or 1
C. 1 or 2
D. -1 or -2

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15. The equation $x^2 + 4x + c = 0$ has no real roots. Which of the following must be true?

I. $c > 4$ II. $c > 0$ III. $x^2 + 4x + (c+1) = 0$ has no real roots

A. I only
B. I and II only
C. I and III only
D. I, II and III

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16. Which of the following equations has two distinct real roots?

A. $x^2 + 9 = 6x$
B. $x^2 + x + 1 = 0$
C. $(x-1)^2 = 2$
D. $x(x-2) = -2$

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17. If the equation $x^2 - 4x + (k-1) = 0$ has real roots, then

A. $k < 5$
B. $k \le 5$
C. $k \ge 5$
D. $k > 5$

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18. If the equation $kx^2 + 4x + 1 = 0$, where $k \neq 0$, has no real roots, then

A. $k < 4$
B. $k > -4$
C. $k > 4$
D. $k < -4$

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19. The equation $x^2 + bx + c = 0$ has two equal real roots. Which of the following must be true?

I. $b^2 = 4c$ II. $c \ge 0$ III. The root of the equation is $\dfrac{b}{2}$

A. I only
B. I and II only
C. I and III only
D. I, II and III

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20. How many distinct real roots does the equation $(x^2 - 4x + 4)(x^2 + 4x + 5) = 0$ have?

A. 1
B. 2
C. 3
D. 4

Next lesson: Graphs of quadratic functions. We will see what Δ means on a graph: how many times $y = ax^2 + bx + c$ meets the x-axis.

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